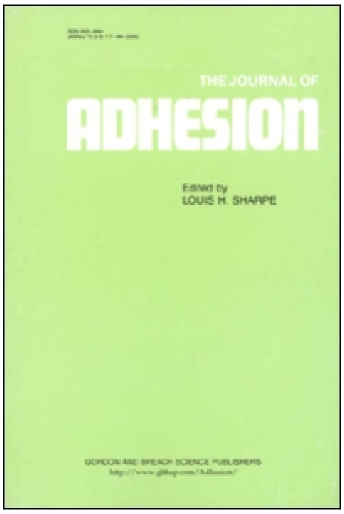

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Stresses in Adhesive between Dissimilar Adherends

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Stresses in Adhesive between Dissimilar Adherends

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The homogeneous equations for the stresses in an adhesive layer between dissimilar adherends are integrated. The normal and shear stresses and some of their space derivatives are defined as state variables, and closed-form expressions are obtained for each. An example is given.

INTRODUCTION

One of the important applications of high modulus graphite fiber-epoxy composities relates to their use as a stiffening agent in a structure. For reasons of economics or proposed use, it may not be feasible, if indeed possible, to build a functional structure entirely of graphite fiber composites. On the other hand, significant weight and cost savings may be realized if graphite fibers are used judiciously in conjunction with other structural materials.

The reinforcing of a structure with graphite fibers will in many cases require that the fiber–epoxy composite be attached to the structure by means of an adhesive layer. Thus, to a great extent, the behavior and the integrity of the stiffened structure will depend largely on the ability of the adhesive layer to transmit the various components of stress.

This work is an attempt to determine theoretically the stresses in an adhesive layer when the joint is continuous and is principally in bending. This problem arises quite naturally when beam-type structures are stiffened by the addition of a layer of stiff material.[1]

The elastic continuum model is the same as that used by Goland and Reissner[2] to determine stresses in cemented lap joints in cases where the joint flexibility is due mainly to that of the adhesive layer. This model should be appropriate for joints occurring in metal structures stiffened by fiber–epoxy composites.

DERIVATION OF STRESS EQUATIONS

The loaded structure with the sign conventions is shown in Figure 1. The definition of the problem suggests that the upper adherend behaves as a plate of bending rigidity $D_1 = E_1t_1^3/[12(1 - \nu_1^2)]$ and the lower adherend behaves as a beam of bending rigidity E_2I_2. The subscripts 1 and 2 denote quantities relating to the upper and lower adherends, respectively, and ν is Poisson's ratio. E is the modulus of elasticity and I is the area moment of inertia. The bending moments, the vertical shear and the axial tension in the adherends are defined by M, V and T, respectively.

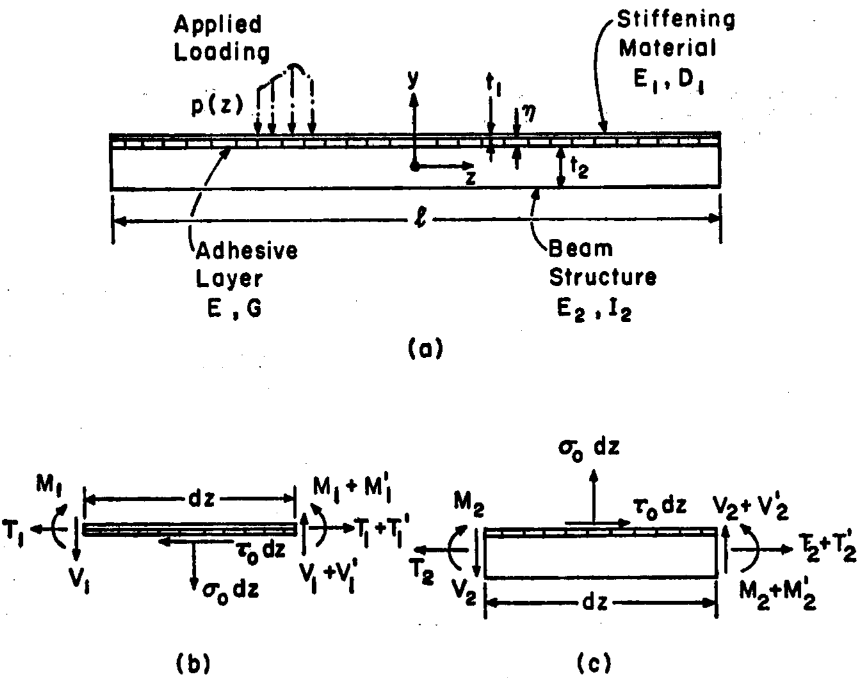

FIGURE 1 Definition of problem showing (a) entire structure, (b) upper layer and adhesive and (c) lower layer and adhesive.

The adhesive layer which is relatively flexible is assumed to be so thin that its bending rigidity can be neglected. The transverse normal (tearing) stress and the shear stress in the adhesive are denoted as σ_0 and τ_0, respectively.

In general, it is not difficult to find a particular solution that will satisfy the nonhomogeneous equations that govern the adhesive stresses. However, if such a particular solution does not automatically satisfy the existing boundary conditions, the required homogeneous solution will likely be significantly more time-consuming. For the problem considered here, all externally applied surface loading functions $p(z)$ will be defined as zero, thus resulting in a set of homogeneous differential equations.

The requirements for moment equilibrium are

$$\begin{aligned} -M_1' - V_1 + \tau_0 \tfrac{1}{2} t_1 &= 0 \\ -M_2' - V_2 + \tau_0 \tfrac{1}{2} t_2 &= 0 \end{aligned} \tag{1}$$

where $(\ )'$ denotes $\dfrac{\partial}{\partial z}(\ )$.

The requirements for horizontal force equilibrium are

$$\begin{aligned} T_1' - \tau_0 &= 0 \\ T_2' + \tau_0 &= 0. \end{aligned} \tag{2}$$

The requirements for vertical force equilibrium are

$$\begin{aligned} V_1' - \sigma_0 &= 0 \\ V_2' + \sigma_0 &= 0. \end{aligned} \tag{3}$$

From plate theory and beam theory

$$\begin{aligned} v_1'' &= \frac{M_1}{D_1} \\ v_2'' &= \frac{M_2}{E_2 I_2} \end{aligned} \tag{4}$$

where v_i are the transverse deflections. If u_1 and u_2 are the corresponding longitudinal displacements of the plate and the beam at the boundaries adjacent to the adhesive, the adherends' strain–stress relations give

$$\begin{aligned} u_1' &= \frac{1}{E_1}\left(\frac{T_1}{t_1} + 6\frac{M_1}{t_1^2}\right) \\ u_2' &= \frac{1}{E_2}\left(\frac{T_2}{t_2} - 6\frac{M_2}{t_2^2}\right). \end{aligned} \tag{5}$$

The elastic stress–strain relations of the adhesive of thickness η are

$$\begin{aligned} \frac{\tau_0}{G} &= \frac{u_1 - u_2}{\eta} \\ \frac{\sigma_0}{E} &= \frac{v_1 - v_2}{\eta}. \end{aligned} \tag{6}$$

The unsubscripted E, G and ν represent the mechanical properties of the adhesive.

The system of equations (1)–(6) can be shown[2] to reduce to

$$\frac{\eta}{G}\tau_0''' = \frac{1}{E_1}\left[\frac{4}{t_1}\tau_0' - \frac{6}{t_1^2}\sigma_0\right] - \frac{1}{E_2}\left[-\frac{4}{t_2}\tau_0' - \frac{6}{t_2^2}\sigma_0\right]$$

and (7)

$$\frac{1}{E}\sigma_0'''' = \frac{1}{\eta}\left[\frac{1}{D_1}\left(\frac{t_1}{2}\tau_0' - \sigma_0\right) - \frac{1}{E_2 I_2}\left(\frac{t_2}{2}\tau_0' + \sigma_0\right)\right].$$

STATE VARIABLE FORMULATION OF EQUATIONS

If a state vector $\mathbf{x}$ is defined as

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{Bmatrix} = \begin{Bmatrix} \tau_0 \\ \tau_0' \\ \tau_0'' \\ \sigma_0 \\ \sigma_0' \\ \sigma_0'' \\ \sigma_0''' \end{Bmatrix}, \tag{8}$$

Eq. (7) may be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{9}$$

or

$$\frac{d}{dz}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & b_2 & 0 & -b_1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \tag{10}$$

where a_1, a_2, b_1 and b_2 are constants† that follow directly from Eq. (7). Assuming that

$$\mathbf{x} = \mathbf{A}e^{rz} \tag{11}$$

and substituting for $\mathbf{x}$ in Eq. (9), the auxiliary equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0‡$$

is.[3]

$$r[r^6 - a_2 r^4 + b_1 r^2 - (a_1 b_2 + a_2 b_1)] = 0 \tag{12}$$

The solutions of Eq. (12) are the eigenvalues of the matrix $\mathbf{A}$.

† These constants will be defined explicitly in the final equations of the solution.

‡ "det" denotes "the determinant of" and $\mathbf{I}$ is the identity matrix.

From Eq. (11), $\mathbf{x}' = r\mathbf{x}$, so for each root r_j

$$
\begin{aligned}
0 &= -r_jA_{j1} + A_{j2} \\
0 &= \quad - r_jA_{j2} + A_{j3} \\
0 &= \quad + a_2A_{j2} - r_jA_{j3} + a_1A_{j4} \\
0 &= \quad - r_jA_{j4} + A_{j5} \\
0 &= \quad - r_jA_{j5} + A_{j6} \\
0 &= \quad - r_jA_{j6} + A_{j7} \\
0 &= \quad + b_2A_{j2} - b_1A_{j4} - r_jA_{j7}.
\end{aligned}
\tag{13}
$$

Eq. (13) define the A_{jk} to within a multiplicative constant. Thus, the general solution is

$$
\mathbf{x} = c_j \begin{Bmatrix} 1 \\ A_{j2} \\ A_{j3} \\ A_{j4} \\ A_{j5} \\ A_{j6} \\ A_{j7} \end{Bmatrix} e^{r_jz}. \tag{14}
$$

where A_{j1} has been arbitrarily set equal to unity.

It can be shown[4] that for all choices of physical parameters, four of the roots are complex conjugate pairs, two of the roots are real, and the seventh is equal to zero. Thus, it can be seen from Eq. (13) that some of the A_{jk} will be complex also. The details of the calculations for the r_j and A_{jk} are straightforward, but lengthy, and are given in reference (4).

COMPLETE SOLUTION

The expanded solution (14) which is in complex form is more useful when expressed exclusively in terms of the real variables of the problem. Thus, the homogeneous solution may be written as

$$
\begin{aligned}
x_j = {} & K_1A_{j1}^0 e^{r_1^0z} + K_2A_{j2}^0 e^{r_2^0z} \\
& + K_3[A_{j3}^0 \cos(r_3^*z) - A_{j3}^* \sin(r_3^*z)]\, e^{r_3^0z} \\
& + K_4[A_{j4}^0 \sin(-r_4^*z) - A_{j4}^* \cos(-r_4^*z)]\, e^{r_4^0z} \\
& + K_5[A_{j5}^0 \cos(r_5^*z) - A_{j5}^* \sin(r_5^*z)]\, e^{r_5^0z} \\
& + K_6[A_{j6}^0 \sin(-r_6^*z) - A_{j6}^* \cos(-r_6^*z)]\, e^{r_6^0z} \\
& + K_7A_{j7}\, e^{r_7^0z}
\end{aligned}
\tag{15}
$$

where $j = 1, 2 \ldots 7$. The K_1 are the constants to be determined in accordance with the boundary conditions. The other terms in Eq. (15) are defined as follows:

$$
a_1 = \frac{6G}{\eta}\left(\frac{1}{E_2t_2^2} - \frac{1}{E_1t_1^2}\right)
$$

$$a_2 = \frac{4G}{\eta}\left(\frac{1}{E_1 t_1} + \frac{1}{E_2 t_2}\right)$$

$$b_1 = \frac{E}{\eta}\left(\frac{1}{D_1} + \frac{1}{E_2 I_2}\right)$$

$$b_2 = \frac{E}{2\eta}\left(\frac{t_1}{D_1} - \frac{t_2}{E_2 I_2}\right)$$

$$p = -a_2$$

$$q = b_1$$

$$s = -(a_1 b_2 + a_2 b_1)$$

$$a = \tfrac{1}{3}(3q - p^2)$$

$$b = \tfrac{1}{27}(2p^3 - 9pq + 27s)$$

$$A = [-\tfrac{1}{2}b + [\tfrac{1}{4}b^2 + \tfrac{1}{27}a_3]^{\frac{1}{2}}]^{\frac{1}{3}}$$

$$B = [-\tfrac{1}{2}b + [\tfrac{1}{4}b^2 + \tfrac{1}{27}a_3]^{\frac{1}{2}}]^{\frac{1}{3}}$$

$$\xi_1 = A + B - \tfrac{1}{3}p$$

$$\xi^* = \tfrac{1}{2}(A - B)(3)^{\frac{1}{2}}$$

$$\xi^0 = -\tfrac{1}{2}(A + B) - \tfrac{1}{3}p$$

$$\xi = [(\xi^0)^2 + (\xi^*)^2]^{\frac{1}{2}}$$

$$\theta_2 = \tan^{-1}(\xi^*/\xi^0)$$

$$\theta_3 = -\tan^{-1}(\xi^*/\xi^0)$$

$$F_3 = \frac{1}{a_1}[\xi^{\frac{3}{2}}\cos(\tfrac{3}{2}\theta_2) - a_2(\xi)^{\frac{1}{2}}\cos(\tfrac{1}{2}\theta_2)]$$

$$F_4 = \frac{1}{a_1}[\xi^{\frac{3}{2}}\cos(\tfrac{3}{2}\theta_3) - a_2(\xi)^{\frac{1}{2}}\cos(\tfrac{1}{2}\theta_3)]$$

$$G_3 = \frac{1}{a_1}[\xi^{\frac{3}{2}}\sin(\tfrac{3}{2}\theta_2) - a_2(\xi)^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta_2)]$$

$$G_4 = \frac{1}{a_1}[\xi^{\frac{3}{2}}\sin(\tfrac{3}{2}\theta_3) - a_2(\xi)^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta_3)]$$

$$M = [F_3^2 + G_3^2]^{\frac{1}{2}}$$

$$N = [F_4^2 + G_4^2]^{\frac{1}{2}}$$

$$\beta_3 = \tan^{-1}(G_3/F_3)$$

$$\beta_4 = \tan^{-1}(G_4/F_4)$$

$$r_1^{\circ} = (\xi_1)^{\frac{1}{2}}$$

$$r_1^* = 0$$

$$r_2^{\circ} = -(\xi_1)^{\frac{1}{2}}$$

$$r_2^* = 0$$

$$r_3^\circ = (\xi)^{\frac{1}{2}}\cos(\tfrac{1}{2}\theta_2)$$
$$r_3^* = (\xi)^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta_2)$$
$$r_4^\circ = (\xi)^{\frac{1}{2}}\cos(\tfrac{1}{2}\theta_3)$$
$$r_4^* = (\xi)^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta_3)$$
$$r_5^\circ = -(\xi)^{\frac{1}{2}}\cos(\tfrac{1}{2}\theta_3)$$
$$r_5^* = -\ (\xi)^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta_3)$$
$$r_6^\circ = -(\xi)^{\frac{1}{2}}\cos(\tfrac{1}{2}\theta_2)$$
$$r_6^* = -(\xi)^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta_2)$$
$$r_7^\circ = 0$$
$$r_7^* = 0$$
$$A_{j1}^0 = 1, \qquad A_{j1}^* = 0 \qquad\qquad j = 1, 2 \ldots 7$$
$$A_{m2}^0 = r_m^0 A_{m1}^0 \qquad\qquad m = 1, 2$$
$$A_{m3}^0 = r_m^0 A_{m2}^0 \qquad\qquad m = 1, 2$$
$$A_{m4}^0 = \frac{1}{a_1}(r_m^0 A_{m3}^0 - a_2 A_{m2}^0) \qquad m = 1, 2$$
$$A_{m5}^0 = r_m^0 A_{m4}^0 \qquad\qquad m = 1, 2$$
$$A_{m6}^0 = r_m^0 A_{m5}^0 \qquad\qquad m = 1, 2$$
$$A_{m7}^0 = r_m^0 A_{m6}^0 \qquad\qquad m = 1, 2$$
$$A_{mn}^* = 0 \qquad\qquad m = 1, 2,\ n = 1, 2 \ldots 7$$
$$A_{32}^\circ = (\xi)^{\frac{1}{2}}\cos(\tfrac{1}{2}\theta_2)$$
$$A_{32}^* = (\xi)^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta_2)$$
$$A_{33}^\circ = \xi\cos(\theta_2)$$
$$A_{33}^* = \xi\sin(\theta_2)$$
$$A_{34}^\circ = F_3$$
$$A_{34}^* = G_3$$
$$A_{35}^\circ = M(\xi)^{\frac{1}{2}}\cos(\tfrac{1}{2}\theta_2 + \beta_3)$$
$$A_{35}^* = M(\xi)^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta_2 + \beta_3)$$
$$A_{36}^\circ = M\xi\cos(\theta_2 + \beta_3)$$
$$A_{36}^* = M\xi\sin(\theta_2 + \beta_3)$$
$$A_{37}^\circ = M\xi^{\frac{3}{2}}\cos(\tfrac{3}{2}\theta_2 + \beta_3)$$
$$A_{37}^* = M\xi^{\frac{3}{2}}\sin(\tfrac{3}{2}\theta_2 + \beta_3)$$
$$A_{42}^\circ = (\xi)^{\frac{1}{2}}\cos(\tfrac{1}{2}\theta_3)$$
$$A_{42}^* = (\xi)^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta_3)$$
$$A_{43}^\circ = \xi\cos(\theta_3)$$
$$A_{43}^* = \xi\sin(\theta_3)$$

$$A^{\circ}_{44} = F_4$$
$$A^{*}_{44} = G_4$$
$$A^{\circ}_{45} = N(\xi)^{\frac{1}{2}} \cos(\tfrac{1}{2}\theta_3 + \beta_4)$$
$$A^{*}_{45} = N(\xi)^{\frac{1}{2}} \sin(\tfrac{1}{2}\theta_3 + \beta_4)$$
$$A^{\circ}_{46} = N\xi \cos(\theta_3 + \beta_4)$$
$$A^{*}_{46} = N\xi \sin(\theta_3 + \beta_4)$$
$$A^{\circ}_{47} = N\xi^{\frac{3}{2}} \cos(\tfrac{3}{2}\theta_3 + \beta_4)$$
$$A^{*}_{47} = N\xi^{\frac{3}{2}} \sin(\tfrac{3}{2}\theta_3 + \beta_4)$$
$$A^{\circ}_{52} = -(\xi)^{\frac{1}{2}} \cos(\tfrac{1}{2}\theta_3)$$
$$A^{*}_{52} = -(\xi)^{\frac{1}{2}} \sin(\tfrac{1}{2}\theta_3)$$
$$A^{\circ}_{53} = \xi \cos(\theta_3)$$
$$A^{*}_{53} = \xi \sin(\theta_3)$$
$$A^{\circ}_{54} = -F_4$$
$$A^{*}_{54} = -G_4$$
$$A^{\circ}_{55} = N(\xi)^{\frac{1}{2}} \cos(\tfrac{1}{2}\theta_3 + \beta_4)$$
$$A^{*}_{55} = N(\xi)^{\frac{1}{2}} \sin(\tfrac{1}{2}\theta_3 + \beta_4)$$
$$A^{\circ}_{56} = -N\xi \cos(\theta_3 + \beta_4)$$
$$A^{*}_{56} = -N\xi \sin(\theta_3 + \beta_4)$$
$$A^{\circ}_{57} = N\xi^{\frac{3}{2}} \cos(\tfrac{3}{2}\theta_3 + \beta_4)$$
$$A^{*}_{57} = N\xi^{\frac{3}{2}} \sin(\tfrac{3}{2}\theta_3 + \beta_4)$$
$$A^{\circ}_{62} = -(\xi)^{\frac{1}{2}} \cos(\tfrac{1}{2}\theta_2)$$
$$A^{*}_{62} = -(\xi)^{\frac{1}{2}} \sin(\tfrac{1}{2}\theta_2)$$
$$A^{\circ}_{63} = \xi \cos(\theta_2)$$
$$A^{*}_{63} = \xi \sin(\theta_2)$$
$$A^{\circ}_{64} = -F_3$$
$$A^{*}_{64} = -G_3$$
$$A^{\circ}_{65} = M(\xi)^{\frac{1}{2}} \cos(\tfrac{1}{2}\theta_2 + \beta_3)$$
$$A^{*}_{65} = M(\xi)^{\frac{1}{2}} \sin(\tfrac{1}{2}\theta_2 + \beta_3)$$
$$A^{\circ}_{66} = -M\xi \cos(\theta_2 + \beta_3)$$
$$A^{*}_{66} = -M\xi \sin(\theta_2 + \beta_3)$$
$$A^{\circ}_{67} = M\xi^{\frac{3}{2}} \cos(\tfrac{3}{2}\theta_2 + \beta_3)$$
$$A^{*}_{67} = M\xi^{\frac{3}{2}} \sin(\tfrac{3}{2}\theta_2 + \beta_3)$$
$$A^{\circ}_{7n} = 0 \qquad \mathrm{n} = 2, \ldots 7$$
$$A^{*}_{7n} = 0 \qquad \mathrm{n} = 2, \ldots 7$$

AN EXAMPLE

Assume the structure of unit depth shown in Figure 1a is loaded by a point load P acting downward at the midspan ($z = 0$) on the upper surface. Let the ends of the structure be simply supported by appropriate supports on the lower surface. Also, let the mechanical and geometric variables be defined as

$$
\begin{aligned}
E_1 &= 4.5 \times 10^7 \text{ psi} \\
E_2 &= 1.0 \times 10^7 \text{ psi} \\
E &= 5.0 \times 10^5 \text{ psi} \\
\nu_1 &= \nu_2 = 0.3 \\
\nu &= 0.33 \\
G_i &= \frac{E_i}{2(1 + \nu_i)} \\
t_1 &= 1.0 \text{ in} \\
t_2 &= 10.0 \text{ in} \\
\eta &= 0.1 \text{ in} \\
l &= 100.0 \text{ in.} \\
P &= 1.0 \text{ lb/unit depth (acting in negative } y\text{-direction).}
\end{aligned}
$$

The simple supported boundary conditions require that at

$$
z = \pm\tfrac{1}{2}l \begin{cases} M_1 = M_2 = V_1 = v_2 = T_1 = T_2 = 0 \\ V_2 = \pm\tfrac{1}{2}P \end{cases} \tag{16}
$$

Taking into account the portion of the structure defined by $z > 0$, the boundary conditions may be taken as

$$
\begin{aligned}
\text{at } z = 0 \quad & \sigma_0' = 0 \\
& \tau_0 = 0
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\text{at } z = \tfrac{1}{2}l \quad & \sigma_0'' = 0 \\
& \tau_0' = 0 \\
& \sigma_0''' = \frac{E}{\eta}\left[\frac{t_1}{2D_1}\tau_0 - \frac{1}{2E_2I_2}(P + \tau_0 t_2)\right] \\
& \tau_0'' = \frac{3G_c}{\eta}\left[\frac{D_1}{E_1^2 t_1 I_1}\tau_0 + \frac{1}{E_2 t_2^2}(P + \tau_0 t_2)\right]
\end{aligned} \tag{18}
$$

and

$$
\int_0^{\frac{1}{2}l} \sigma_0 \, dz = \tfrac{1}{2}P. \tag{19}
$$

Eq. (17) are statements of symmetry and Eq. (19) is an equilibrium requirement. Conditions (18) were determined by combining Eqs. (1)–(6) and (16).

The normal stress and the shear stress are plotted in Figure 2. The character of the normal stress in the vicinity of $z = 0$ is analogous to the classical beam-on-elastic-foundation response. The rapidly varying boundary-layer character of σ_0 near $z = 0$ is consistent with the concentrated load acting at the center of the composite structure. Also, note that the equilibrium condition defined by Eq. (19) is satisfied. The nearly uniform level of τ_0 is as would be expected as the two structural components undergo relative shearing displacements. The behavior of τ_0 in the vicinity of $z = 0$ and $z = l/2$ is consistent with the imposed boundary conditions.

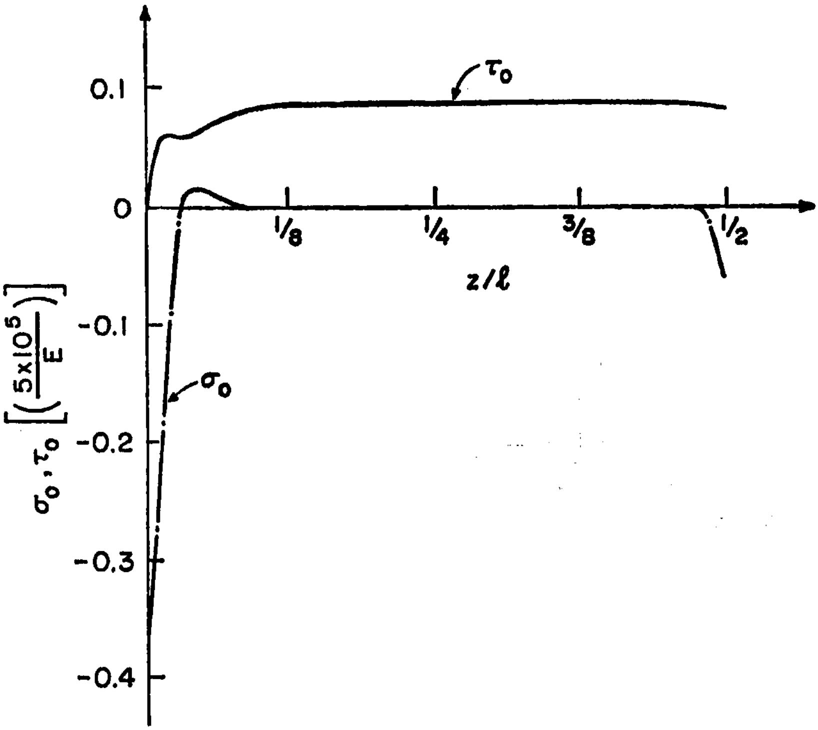

FIGURE 2 Stresses in adhesive layer.

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